DEGREE OF MASTER OF SCIENCE

MATHEMATICAL MODELLING AND SCIENTIFIC COMPUTING

A1 Mathematical Methods I

HILARY TERM 2016

THURSDAY, 14 JANUARY 2016, 9.30am to 11.30am

Candidates should submit answers to a maximum of four questions that include an answer to at least one question in each section.

Please start the answer to each question on a new page.
All questions will carry equal marks.

Do not turn this page until you are told that you may do so

Section A: Applied Partial Differential Equations

1. Consider the PDE

$$au_x + bu_y = c, \quad (x, y) \in D, \tag{1}$$

where a, b, c are smooth functions of x, y, and u.

(a) (i) [6 marks] Suppose data is given along the curve

$$\Gamma : (x_0(s), y_0(s), u_0(s))^T.$$

Show that if the projection of Γ on the (x, y) plane aligns with a characteristic projection at a point, then locally u_x and u_y cannot be uniquely determined.

(ii) [6 marks] Suppose that

$$\alpha a(x, y) + \beta b(x, y) + \gamma c(x, y) \equiv 0 \quad \forall (x, y) \in D$$

for some constants $\alpha, \beta, \gamma \in \mathbb{R}$.
Show that $\alpha x + \beta y + \gamma u$ is constant along characteristics.

(b) In (1), let $a(x, y, u) = x$, $b(x, y, u) = y$, $c(x, y, u) = (x + y)u$.

(i) [5 marks] Show that characteristics meet at a single point for any Cauchy data.

(ii) [8 marks] Given the data $u = 1$ on $x = 1$, $1 < y < 2$, obtain an explicit solution and determine the domain of definition.

2. (a) [9 marks] Convert the following PDE to conservation form and determine the Rankine-Hugoniot condition for a shock across which u jumps from u^- to u^+

$$u^2 u_x + yuu_y = x - \frac{u^2}{2}.$$

Any conservation form is acceptable.

(b) [16 marks] Consider the first order problem for $u(x,t)$, where x represents spatial position and t represents time:

$$u_t + \frac{\partial}{\partial x}\left(u(1-u)\right) = 0, \quad t > 0,$$

$$u(x,0) = \begin{cases} 1/4, & x < 0 \\ b, & x > 0. \end{cases}$$

Find conditions on $b \in (0,1)$ such that

(i) A shock exists that moves forward with time.

(ii) A shock forms that stays at a fixed x location.

(iii) No shock forms.

Sketch the characteristic projections and shock (if one exists) in the (x,t)-plane in the case $b = 3/4$.

3. (a) [12 marks] Derive the PDE and boundary conditions satisfied by the Green's function for the Dirichlet problem

$$u_{xx} + u_{yy} + a(x,y)u_x + b(x,y)u_y + cu = f(x,y), \quad (x,y) \in D$$
$$u = g(x,y), \quad (x,y) \in \partial D$$

where a, b, c, f, g are given smooth functions.
Derive the form of the solution in terms of the Green's function. *You do not need to compute the Green's function.*

(b) [13 marks] Consider the problem

$$\nabla^2 u(x,y) = 0 \quad \text{on } y > 0,$$
$$u(x,0) = u_0(x),$$

where $u_0(x) \to 0$ sufficiently rapidly as $x \to \pm\infty$. Show from the Green's function representation that

$$u(x,y) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} u_0(x + y\tan\theta)\, d\theta.$$

4. (a) Consider the following system of PDEs for $u(x,y)$, $v(x,y)$

$$\begin{aligned}&2x^2yu_x + 5xy^2u_y + 2x^2y^2v_y + 5xyu + x = 0,\\ &yu_y - x^2v_x + u - 2xv = 0.\end{aligned}$$

(i) [8 marks] In the region $x > 0, y > 0$, determine whether the system is hyperbolic, elliptic, or parabolic.

(ii) [9 marks] Show that the system is equivalent to a second order semilinear PDE.

(b) Determine the characteristic coordinates $\xi(x,y)$ and $\eta(x,y)$ for the PDE

$$xu_{xx} + u_{yy} = 0$$

in the regions

(i) [4 marks] $x < 0$,

(ii) [4 marks] $x > 0$.

Section B: Supplementary Mathematical Methods

5. The differential operator L is defined by

$$Ly \equiv x^2 y''(x) - 5xy'(x) + 9y(x)$$

on $1 < x < \mathrm{e}^{\pi}$.

(a) [11 marks] You are given that

$$\lambda_k = -k^2, \quad k = 0, 1, 2, \ldots$$

are (all) the eigenvalues of $Ly_k = \lambda_k y_k$ for the boundary conditions

$$y(1) - y'(1) = 0, \quad y(\mathrm{e}^{\pi}) - \mathrm{e}^{\pi} y'(\mathrm{e}^{\pi}) = 0; \tag{2}$$

find the corresponding eigenfunctions.

(b) (i) [5 marks] Find an appropriate function $r(x)$ so that $\hat{L} \equiv rL$ is a Sturm-Liouville operator. Give the eigenvalues $\hat{\lambda}$ and eigenfunctions $\hat{y}$ for $\hat{L}\hat{y} = \hat{\lambda} r \hat{y}$ with boundary conditions

$$\hat{y}(1) - \hat{y}'(1) = 0, \quad \hat{y}(\mathrm{e}^{\pi}) - \mathrm{e}^{\pi} \hat{y}'(\mathrm{e}^{\pi}) = 0.$$

(ii) [9 marks] Use this to obtain a formula for the coefficients in an eigenfunction expansion

$$y = \sum_{n=0}^{\infty} c_k \hat{y}_k(x)$$

for the solution of the problem

$$x^2 y''(x) - 5xy'(x) + 9y(x) + by(x) = e^{2x},$$

with the boundary conditions (2), where $b < 0$ is a real constant.

You do not need to evaluate the integrals in the formula for c_k.

6. (a) (i) [6 marks] Find the general **real** solution of the linear differential equation:

$$Ly \equiv \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0, \qquad (3)$$

for $0 < x < \pi$.

(ii) [6 marks] Consider the boundary value problem

$$Ly(x) = f(x), \quad 0 < x < \pi, \qquad y(0) = 0, \quad y(\pi) + \frac{\mathrm{d}y}{\mathrm{d}x}(\pi) = 0, \qquad (4)$$

with Ly as in (3). Write down two equivalent problems for the Green's function $g(x, \xi)$:
(I) using the delta function $\delta(x)$;
(II) using only classical functions and with appropriate conditions at $x = \xi$.
Determine $g(x, \xi)$ explicitly.

(b) The set C_0^∞ denotes the set of "test" functions, i.e. all functions that have compact support and have derivatives of arbitrary order.

(i) [5 marks] Define what it means to say that the map $T : C_0^\infty \to \mathbb{R}$ is a *distribution.*

(ii) [8 marks] Let $H(x)$ denote the Heaviside function. Show that

$$(x^2 H(x))''' = \alpha\delta(x) + \beta\delta'(x)$$

in the distributional sense, with constants α and β that you should determine.